Unit – 1

1. **Differential Equation :-** An equation containing derivative of one or more independent variables w.r. to one or more independent variables is called differential Equation.

D.E → ODES, PDES

Example :- (1) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$, (3). $y\frac{\partial^2 z}{\partial x^2} + e^{z} = 0$

(2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, (4) $y\frac{\partial^2 z}{\partial x^2} + e^{\frac{\partial z}{\partial x}} = 0$

2. **Notation :** Consider $z = f(x,y)$ ——— (1)

then
$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2},$$
$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

3. **Order of Partial D.E. =** Order of PDE is the order of the highest derivatives & coefficient of present in PDE's

4. **degree of PDE :** Degree of a partial differential equation is the ~~big~~ power of highest ordered derivative present in the equation when it has made free from radical and fractional power.

5. **Solution of PDEs :-** The general solution of PDE contains arbitrary constant, or arbitrary functions or both. consequently, we can say that PDE can be formed by arbitrary constt. or arbitrary fuction.

Formation of PDE

Arbitrary Constt

Consider $f(x,y,z,a,b)=0$ —— (1)

$a,b \to$ constant

$x,y \to$ independent variable.

z - dependent variable on x & y. (i.e. $z=g(x,y)$)

Partial diff (1) w.r. to x

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \quad \text{—— (2)}$$

Partial diff (1) w.r. to y

$$\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \quad \text{—— (2)}$$

Using (1), (2) & (3) Eliminating a & b then

$$\boxed{f(x,y,z,p,q)=0}$$

it is called PDE.

Example:- Form PDE from the following equations by eliminating the arbitrary Constt.

(i). $z=ax+by+ab.$

(ii) $z=(x+a)(y+b)$

(iii) $az+b=a^2x+y$

Arbitrary funtion

Consider $\phi(u,v)=0$ —— (1)

where u & v are funtion of x,y & z and z is funtion of in terms of x & y. and ϕ is called arbitrary funtion.

Diff (1) w.r. to x then (Using chain Rule)

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0.$$

$$\frac{\partial\phi/\partial u}{\partial\phi/\partial v}=-\left(\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}\right) \quad \text{—— (2)}$$

Similary (Diff w.r. to y)

$$\frac{\partial\phi/\partial u}{\partial\phi/\partial v}=-\left(\frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}\right) \quad \text{—— (3)}.$$

from (2) & (3).

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\boxed{Pp+Qq=R}$$

which is called PDE of First degree in p and q.

Example:-

$z=f(x^2-y^2)$

$z=\phi(x)\cdot\psi(y)$

$z=f(x+it)+g(x-it).$

Complete Solution :- and general solution : The solution $f(x, y, z, a, b) = 0$ of a I^{st} order PDE, which two arbitrary constt is called a Complete solution. ③

Put $b = \phi(a)$ then.

$f(x, y, z, a, \phi(a)) = 0$, we get a solution involving an arbitrary ~~constt~~ funtion

This is called general solution

Partiallar solution :- A solution obtained from the complete solution by giving particular values to the arbitrary constt.

Types of PDE :-

L.P.DE | Non LPDE | Semi LPDE | quasi LPDE

Homogeneous :- A linear PDE with constant coefficient in which all the partial derivative are same order.

① APDE is said to LPDE if it of first degree in the dependent variable and its partial derivatives and they are not multiplied together.

① $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$, ② $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy$

② An equation which is not L.PDE is called non-L.PDE equation

ⓐ $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} = 1$ ⓑ $\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial^2 u}{\partial x^2} = 0$

③ Semi-LPDE An equation of the form $P(x,y)p + Q(x,y)q = R(x,y,z)$ is called semi-LPDE Ex. $\boxed{xyp + yx^2q = xz^2}$

④. quasi-linear PDE: An equation of the form $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ is called quasi linear PDE

$x^2zp + yzq = xy\sin z$

Equation Solvables by Direct integration :- {Containing only one partial Derivative}

(4)

Example:- Solve $t = \sin xy$

$$\frac{\partial^2 z}{\partial y^2} = \sin(xy)$$

Integrating w.r. to y. then $\frac{\partial z}{\partial y} = -\frac{1}{x}\cos(xy) + f(x)$

Again.

$$\boxed{z = -\frac{1}{x^2}\sin(xy) + y\,f(x) + \phi(x)}$$

Lagrange's Equation:-

Quasi - linear PDE

Consider $Pp + Qq = R$ — ① where P, Q & R are functions of x, y and z.

Lagrange's Auxiliary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Then the solution of PDE $Pp + Qq = R$ as $\phi(u,v) = 0$ ~~and~~ or $v = f(u)$ or $u = f(v)$. where ϕ is arbitrary function of $u(x,y,z)$ & $v(x,y,z)$ {Here $c_1 = u$, $c_2 = v$}

Working Rule :-

Step I find P, Q and R. {Using $Pp + Qq = R$}

Step II Form the Auxiliary Equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step III Solve Auxiliary Equation by grouping method or Multipliers or both.

Step IV :. then $\phi(u,v) = 0$ or $v = f(u)$ or $u = f(v)$ is general solution of the equation $Pp + Qq = R$.

e.g. solve $xp + yq = z$.

Example: 1 Solve $y^2 p - xyq = x(z-2y)$

Solution: The given equation $y^2 p - xyq = x(z-2y)$ ——— ①.

then $P = y^2$, $Q = -xy$ and $R = x(z-2y)$.

Auxiliary Equation

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

I II III

Taking I & II fraction

$$\frac{dx}{y} = \frac{dy}{-x} \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 = c_1$$

Taking II & III fraction

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \Rightarrow y^2 = yz + c_1.$$

general solution is

$\phi(x^2+y^2, y^2-yz) = 0$

Q2 Solve $x^2 p + y^2 q = (x+y)z$, —— (1)

Compare $Pp + Qq = R$ then

$P = x^2$, $Q = y^2$ & $R = (x+y)z$

A.E

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad \text{—— (2)}$$

from (I) (II) (III)

I & II fraction

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} - c_1 \Rightarrow \frac{1}{x} - \frac{1}{y} = c_1$$

& from (2).

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx-dy}{\cancel{(x+y)}(x-y)} = \frac{dz}{\cancel{(x+y)}z}$$

$$\log(x-y) = \log z + \log c_2 \Rightarrow \frac{x-y}{z} = c_2$$

Hence the general solution

$$\boxed{\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0}$$

3. Solve $p + 3q = 5z + \tan(y-3x)$ —— (1)

A.E.

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

I II III

$y - 3x = c_1$ and I & III $\dfrac{dx}{1} = \dfrac{dz}{5z + \tan c_1}$

$$\Rightarrow x = \frac{1}{5}\log(5z + \tan c_1) - \frac{c_2}{5}$$

$$\log\{5z + \tan(y-3x)\} - 5x = c_2$$

Hence general solution

$$\phi(y-3x, \log(5z + \tan(y-3x)) - 5x) = 0..$$

Q: Solve $(y^2+z^2)p - xyq = -zx$

Lagranges A.E

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

From II & III fractions.

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\frac{y}{z} = c_1$$

Now Using multipliers as x, y, z we get

each fraction $= \frac{x\,dx + y\,dy + z\,dz}{0}$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$\phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$$

Non-linear PDE of First order:- A partial DE which ⓐ is involves I^{st} order

partial derivative p and q with degree higher than one and the product of p and q is called a non-linear PDE. of the first order.

Case I Equation of the form $f(P, q) = 0$.

In this case, let $p = a$. (constant)

and solve equation for q. (find q).

Put in $\boxed{dz = p\,dx + q\,dy}$ and solve it.

Example 1. Solve $pq = p + q$

Sol^n. The given equation $pq = p + q$ —— ①

let $p = a$ then. equation solve for a

$$aq = a + q \Rightarrow q = \frac{a}{a-1}$$

Now $dz = a\,dx + \left(\frac{a}{a-1}\right) dy$

Integration

$$\boxed{z = ax + \left(\frac{a}{a-1}\right) y + c}$$

Example: 2. Solve $x^2 p^2 + y^2 q^2 = z^2$

Sol^n:- The given equation $x^2 p^2 + y^2 q^2 = z^2$

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{—— ①}$$

Let $\frac{\partial x}{x} = \partial X$, $\frac{\partial y}{y} = \partial Y$, $\frac{\partial z}{z} = \partial Z$

then $X = \log x$, $Y = \log y$, $Z = \log z$

Now, $\frac{\partial Z}{\partial X} = \frac{x}{z}\frac{\partial z}{\partial x}$, $\frac{\partial Z}{\partial Y} = \frac{y}{z}\cdot\frac{\partial z}{\partial y}$ where $P = \frac{\partial Z}{\partial X}$

$Q = \frac{\partial Z}{\partial Y}$

Equation ①

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \Rightarrow P^2 + Q^2 = 1$$

It is of the form $f(P, Q) = 0$

Let $p = a$ the $q = \sqrt{1-a^2}$

thus. $dz = P\,dx + Q\,dY$

$$z = ax + \sqrt{1-a^2}\,y + c$$

$$\boxed{\log z = a \log x + \sqrt{1-a^2} \log y + c}$$

Case II. Equation of the form $f(z, p, q) = 0$.

(Note:- Equation not containing x and y).

Let $\boxed{p = aq}$

Q. 1. Solve $z^2(p^2+q^2+1) = a^2$, the equation in form $f(z, p, q) = 0$

Put $p = aq$. then,

$$z^2(a^2q^2+q^2+1) = a^2$$

$$q^2(a^2+1)+1 = \frac{a^2}{z^2}$$

$$q = \pm\sqrt{\left(\frac{1}{a^2+1}\right)\left(\frac{a^2-z^2}{z^2}\right)}$$

Thus,

$$dz = p\,dx + q\,dy$$

$$dz = aq\,dx + \sqrt{\left(\frac{1}{a^2+1}\right)\left(\frac{a^2-z^2}{z^2}\right)}\,dy$$

$$dz = \pm\sqrt{\left(\frac{a}{a^2+1}\right)\left(\frac{a^2-z^2}{z^2}\right)}\,dx \pm \sqrt{\left(\frac{1}{a^2+1}\right)\left(\frac{a^2-z^2}{z^2}\right)}\,dy$$

$$\pm\sqrt{a^2+1}\int\frac{z}{\sqrt{a^2-z^2}}\,dz = ax + y + c$$

$$\pm\sqrt{1+a^2}\sqrt{a^2-z^2} = ax + y + c$$

$$\boxed{(1+a^2)(a^2-z^2) = (ax+y+c)^2}$$

Case III Equations of the form $f_1(x, p) = f_2(y, q)$

i.e; equation in which z is absent and the terms involving x and p can be separated from those involving y and q.

Example: solve $p^2 - q^2 = x - y$

$$p^2 - x = q^2 - y = a \text{ (constant)}.$$

Now it is form $f(x, p) = f(y, q)$.

$$p^2 = x + a, \quad q^2 = y + a$$

$$p = \sqrt{x+a}, \quad q = \sqrt{y+a}$$

Putting p and q in $dz = p\,dx + q\,dy$.

$$dz = (\sqrt{x+a})\,dx + (\sqrt{y+a})\,dy$$

$$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c}$$

Case IV Clairaut Equation

Equation in form.. $z = px + qy + f(p, q)$

The Complete solution is $z = ax + by + f(a, b)$

obtained by writing a for p & b for q.

Example:- Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$

$$\boxed{z = ax + by + \sqrt{1 + a^2 + b^2}}$$

Example:- Solve $4xyz = pq + 2px^2y + 2qxy^2$.

Let $x^2 = X$ & $y^2 = Y$, $p = 2x\frac{\partial z}{\partial X}$, $q = 2y\frac{\partial z}{\partial Y}$

$$\cancel{4xy}z = \cancel{4xy}\frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y} + \cancel{4}x^{\cancel{3}\,2}\cancel{y}\frac{\partial z}{\partial X} + \cancel{4x}y^{\cancel{3}\,2}\frac{\partial z}{\partial Y}$$

$$z = x^2\frac{\partial z}{\partial X} + y^2\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}$$

Thus. Complete solution is $z = aX + bY + ab = ax^2 + by^2 + ab$.

Example: Solve $z^2(p^2x^2+q^2)=1$. (11)

Charpit's Method:-

This is a general method for finding the complete solution of non-linear PDE of first order.

Standard form $f(x,y,z,p,q)=0$

Charpit's Auxiliary Equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Using these two member

Examples:- Solve $2zx-px^2-2pxy+pq=0$ —— (1)

Sol^n:- we have $f\equiv 2zx-px^2-2qxy+pq=0$

$$\frac{\partial f}{\partial x}=2z-2px-2qx, \quad \frac{\partial f}{\partial y}=-2qx, \quad \frac{\partial f}{\partial z}=2x$$

$$\frac{\partial f}{\partial p}=-x^2+q, \quad \frac{\partial f}{\partial q}=-2xy+p$$

Charpit's Auxiliary Equation

$$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dz}{+px^2-2pq+2qxy}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dF}{0}$$

Now $dq=0 \Rightarrow q=a$

Putting $q=a$ into (1) then

$$2zx-px^2-2axy+ap=0$$

$$p=\frac{2x(z-ay)}{x^2-a}$$

Putting p and q into $dz=p\,dx+q\,dy$

$$dz=\frac{2x(z-ay)}{x^2-a}dx+a\,dy$$

$$\frac{d(z-ay)}{z-ay}=\frac{2x}{x^2-a}dx \Rightarrow \log(z-ay)=\log(x^2-a)+\log b$$

$$\Rightarrow z=b(x^2-a)+ay.$$

Q.2 Solve $(p^2+q^2)y = qz$, $\Rightarrow f \equiv (p^2+q^2)y - qz$ —— (1)

Sol^n: $f_x = 0$ $f_y = p^2+q^2$, $f_z = -q$, $f_p = 2py$, $f_q = 2qy$

Charpit's auxiliary equation

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

Taking I & II fraction

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p^2+q^2 = a^2 \text{ —— (2)}$$

from (1) $q = \dfrac{a^2y}{z}$

from (2)

$$p = \frac{a}{z}\sqrt{z^2-a^2y^2}$$

putting p and q into $dz = pdx + qdy$

$$dz = \frac{a}{z}\sqrt{z^2-a^2y}\,dx + \frac{a^2y}{z}dy.$$

$$\frac{z\left(dz - \frac{a^2y}{z}dy\right)}{\sqrt{z^2-a^2y}} = \cancel{a\,dx} = a\,dx$$

$$zdz - a^2y\,dy = a\,dx$$

$$\sqrt{z^2-a^2y} = ax+b$$

$$\boxed{z^2 = (ax+b)^2 + a^2y^2}$$

Cauchy's Method of Characteristics:-

Standard form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x,y) + ku; \quad u(0,y) = h(y). \quad \text{--- (1)}$$

Let $u(x,y)$ be the solution of (1)

then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \quad \text{--- (2)}$$

from (1) & (2) then

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y)+ku}$$

Note:- first Constant c & second Constant $g(c)$

Example Using cauchy's method of characteristic to solve the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y \qquad \text{and } u(x,0) = 0.$$

Soln:- The given PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y \quad \text{--- (1)}$$

A.E

$$\underset{(I)}{\frac{dx}{1}} = \underset{(II)}{\frac{dy}{1}} = \underset{(III)}{\frac{du}{x+y}}$$

Taking (I) & (II) fraction

$$dx - dy = 0 \Rightarrow x - y = c \Rightarrow x = c + y.$$

Taking II & III

$$\frac{dx}{1} = \frac{dz}{2y+c} \Rightarrow u = y^2 + cy + g(c)$$

Put $c = x - y$

$$u = y^2 + (x-y)y + g(x-y)$$

$u(x,0) = 0$

$$g(x) = 0 \Rightarrow g(x-y) = 0$$

$$u(x,y) = xy$$

which required solution.

Homogeneous L.P.D.E with Const Coefficient :-

An equation in the form

$$(a_0 D^n + a_1 D^{n-1} \cdot D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n) z = F(x, y) \quad (1)$$

where a's are Constant, order of each terms ~~are~~ must be same. (order n)

& $D \equiv \frac{\partial}{\partial x}$ & $D' \equiv \frac{\partial}{\partial y}$.

from (1) or

$$\phi(D, D') z = F(x, y)$$

Complete solution or general solution

Complementary function (C.f) + Particular integral (P.I).

which is the complete solution of the equation $\phi(D, D') z = 0$ it must contain n arbitrary funtions, where n is the order of DE

A solution obtained by complete solution/general soluti by giving particular values to the arbitary constants.

i.e. solution of equation (1) ~~as~~ is

$$\boxed{Z = C.F + P.I}$$

equation (1) written as:

$$[(D - m_1 D')(D - m_2 D') \dots (D - m_n D')] Z = 0 \quad (3)$$

Finding C.F :- Putting $f(x, y) = 0$, $D = m$ & $D' = 1$ into (1) then

A.E

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

Case I, Distinct Roots, $m_1, m_2, \dots m_n$ are not equal

then Complementary funtion ~~of~~ (1)

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$$

Case II Repeated roots, $m_1 = m_2 = m_3, m_4, \dots m_{n-3}$

the $C.f = f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots + f_{n-3}(y + m_{n-3} x)$.

Homogeneous L.P.D.E with constt Coefficient :-

An equation in the form.

$$(a_0 D^n + a_1 D^{n-1} \cdot D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = F(x,y) \quad \text{—(1)}$$

where a's are Constant, order of each terms ~~are~~ must be same. (order n)

& $D \equiv \frac{\partial}{\partial x}$ & $D' \equiv \frac{\partial}{\partial y}$.

from (1) or

$$\phi(D, D') z = F(x,y)$$

$\boxed{\text{Complete solution}}$ or $\boxed{\text{general solution}}$

$\boxed{\text{Complementary function}}$ (C.f) + $\boxed{\text{Particular integral}}$ (P.I).

Complementary function (C.f): which is the complete solution of the equation $\phi(D,D')z = 0$ it must contain n arbitrary functions, where n is the order of DE

Particular integral (P.I): A solution obtained by complete solution/general solution by giving particular values to the arbitrary constants.

i.e. Solution of equation (1) ~~eq~~ is

$$\boxed{Z = C.F + P.I}$$

equation (1) written as:

$$[(D - m_1 D')(D - m_2 D') \cdots (D - m_n D')] Z = 0 \quad \text{—(3)}$$

Finding C.F :- Putting $f(x,y) = 0$, $D = m$ & $D' = 1$ into (1) then

A.E $\quad a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$

Case I. Distinct Roots, $m_1, m_2, \ldots m_n$ are not equal

then Complementary function ~~of~~ (1)

~~C.F = yc~~ $\quad C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots + f_n(y + m_n x)$

Case II Repeated roots, $m_1 = m_2 = m_3, m_4, \ldots m_{n-3}$

the $\quad C.F = f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \cdots + f_{n-3}(y + m_{n-3} x)$.

Q. Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

$r - 4s + 4t = 0$

Sol: The given equation $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

A.E. $(m^3 - 6m^2 + 11m - 6)\cancel{z} = 0$

$m = 1, 2, 3.$

$C.F = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

$P.I = 0$

general solution ~~complete~~ solution is

$$z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$$

Q. Solve $(D^3D'^2 + D^2D'^3)z = 0$

written as: $D^2D'^2(D+D')z = 0.$

$(D + 0D')^2(D' + 0D)^2(D+D') = z = 0$

Hence general solution is

$$z = \phi_1(x) + x\phi_2(x) + \phi_3(y) + y\phi_4(y) + \phi_5(x-y)$$

Q. Solve $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$

Finding P.I.

Consider $\phi(D,D')z = f(x,y)$, $P.I = \frac{1}{\phi(D,D')} f(x,y)$

Case I, where $f(x,y) = g(ax+by)$.

Case I when $\phi(D,D') \neq 0$. then

$$\frac{1}{\phi(D,D')} g(ax+by) = \frac{1}{\phi(a,b)} \iiint f(u)\,du\,du\,du$$

multiple integral = order of PDE.

Case II when $f(a,b) = 0$ then

$$\frac{1}{(bD - aD')^n} g(ax+by) = \frac{x^n}{b^n n!} g(ax+by).$$

Q. Solve the LPDE $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+y)$

Sol^n The given equation $(D^2+D'^2)z = \cos mx \cos ny + 30(2x+y)$

Auxiliary Equation

$m^2+1=0 \Rightarrow m = \pm i$ $\quad C.F = f_1(y+ix) + f_2(y-ix)$

$$P.I. = \frac{1}{(D^2+D'^2)}\left\{\frac{1}{2}\cos(mx+ny) + \cos(mx-ny)\right]$$

$$+ \frac{1}{(D^2+D'^2)}\{30(2x+y)\}$$

$$= \frac{-1}{2}\left\{\frac{1}{(m^2+n^2)}\cos(mx+ny) - \frac{1}{(m^2+n^2)}\cos(mx+ny)\right.$$

$$+ \frac{30}{2^2+1^2}\left(\frac{1}{6}(2x+y)^3\right)$$

$$= \frac{-1}{2(m^2+n^2)}\left[\cos(mx+ny) + \cos(mx-ny)\right] + (2x+y)^3$$

Q. Solve the LPDE $\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$ (19)

Sol(1):- The given equation is $(D^3 - 3D^2D' + 4D'^3)u = e^{x+2y}$

where $D \equiv \frac{\partial}{\partial x}$

$D' \equiv \frac{\partial}{\partial y}$

Auxiliary Equation is

$$m^3 - 3m^2 + 4 = 0$$

$$m = 2, 2, -1$$

$$C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

&

$$P.I = \frac{1}{(D^3 - 3D^2D' + 4D'^3)} e^{x+2y}$$

$$= \frac{1}{1 - 3\times 2 + 4\times 2^3} \iiint e^u du$$

$$= \frac{1}{27} e^{x+2y}$$

Hence the general solution is

$$u = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

Q.2. Solve the LPDE $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

Sol(2):- The given equation is

$$(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$$

The auxiliary Equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1.$$

$$C.F. = f_1(y+x) + x f_2(y+x)$$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2} \sin(2x+3y)$$

$$= \frac{1}{(2-3)^2} \iint \sin u\, du = -\sin(2x+3y)$$

$$z = f_1(y+x) + f_2(y+x) - \sin(2x+3y).$$

Q. Solve $4r - 4s + t = 16\log(x+2y)$

Sol^{n}:-

$$C.F = f_1(y + \tfrac{1}{2}x) + f_2(y + \tfrac{1}{2}x)$$

$$P.I = \frac{1}{(2D-D')^2} 16\log(x+2y)$$

$$= \frac{x^2}{2^2 \times 2!} 16\log(x+2y)$$

$$= 2x^2\log(x+2y)$$

Q. Solve $r + 2s + t = 2(y-x) + \sin(x-y)$

$$C.F\ f_1(y-x) + xf_2(y-x)$$

$$P.I = \frac{1}{(D+D')^2}\{2(y-x) + \sin(x-y)\}$$

$$= \frac{x^2}{\cancel{4}\,2} \times 2(y-x) + \frac{x^2}{8}\sin(x-y)$$

$$= \frac{x^2}{\cancel{4}}(y-x) + \frac{x^2}{\cancel{8}}\sin(x-y).$$

Q. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

A.E $\cancel{m}^2 - 2 = 0 \Rightarrow \cancel{m} = \pm\sqrt{2}$

$$C.F = f_1(\cancel{y} + \sqrt{2}x) + f_2(y - \sqrt{2}x).$$

$$P.I = \frac{1}{D^2 - 2DD'}\tfrac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

Rule II When $f(x,y)$ is of the form $x^m y^n$

Remarks:- ① if $n<m$, $\frac{1}{f(D,D')}$ should be expanded in power of $\frac{D'}{D}$

② if $n>m$, $\frac{1}{f(D,D')}$ should be expanded in power of D/D'.

Note $\boxed{\text{Binomial}}$ $(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_n x^0 y^n$.

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Q. Solve $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

A.E $m^3 - 1 = 0$

$m = 1, \omega, \omega^2$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

m^2+m+1

$\omega = \frac{-1 \pm \sqrt{-3}}{2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$P.I = \frac{1}{D^3 - D'^3} x^3 y^3$$

$$= \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1}(x^3y^3)$$

$$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3} \dots\right) x^3 y^3$$

$$= \frac{1}{D^3}\left(x^3y^3 + \frac{1}{D^3} 6x^3\right)$$

$$= \frac{1}{D^3}(x^3y^3) + \frac{1}{D^6} 6x^3 = \frac{x^6y^3}{120} + \frac{x^9}{4.5.6.7.8.9}$$

$$= \frac{x^6 y^3}{120} + \frac{x^9}{10080}.$$

Q.2. Solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

General Method To Find the P.I

$$P.I = \frac{1}{\phi(D,D')} f(x,y) = \frac{1}{(D-m_1D')(D-m_2D')\cdots(D-m_nD')} f(x,y).$$

or $$\frac{1}{(D-mD')} f(x,y) = \int \phi(x, c-mx)\, dx.$$

Example:- Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

A.E. $$m^2 + m - 6 = 0$$
$$m = 2, -3$$
$$C.F = f_1(y+2x) + f_2(y-3x)$$

$$P.I = \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$= \frac{1}{(D-2D')} \int (c+3x)\cos x\, dx$$

$$= \frac{1}{(D-2D')} \left[(c+3x)\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'} \left[y\sin x + 3\cos x\right]$$

$$= \int [(b-2x)\sin x + 3\cos x]\, dx$$

$$= -(b-2x)\cos x + 2(\sin x) + 3\sin x.$$

$$= -y\cos x + \sin x \qquad b = y+2x$$

Ex. 4. Solve $r - s - 2t = (2x^2 + xy - y^2)\sin(xy) - \cos xy$

A.E.

$$m^2 - m - 2 = 0 \Rightarrow m = -1, 2$$

$$C.F = f_1(y-x) + f_2(y+2x)$$

$$P.I = \frac{1}{(D+D')(D-2D')}\left[(2x-y)(x+y)\sin xy - \cos xy\right]$$

$$= \frac{1}{(D+D')}\int\left[(2x-c+2x)(c-x)\sin(cx-2x^2) - \cos(cx-2x^2)\right]dx$$

$$= \frac{1}{D+D'}\int\left[(c-x)(4x-c)\sin(cx-2x^2) - \cos(cx-2x^2)\right]dx$$

$$= \frac{1}{(D+D')}(c-x)\cos(cx-2x^2) + \cancel{\int\cos(cx-2x^2)\,dx}$$

$$- \cancel{\int\cos(cx-2x^2)\,dx}$$

$$= \frac{1}{D+D'}(y+x)\cos xy \quad \text{where } c = y+2x$$

$$= \int(b+2x)\cos(bx+x^2)\,dx$$

$$= \sin(bx+x^2) = \sin xy$$

Hence general solution is

$$z = f_1(y-x) + f_2(y+2x) + \sin xy.$$

H.W Solve PDE. $r - t = \tan^3 x \tan y - \tan x \tan^3 y$

Non-Homogeneous Linear PDE with constant coefficients- (28)

Consider $\phi(D, D')z = f(x,y)$ ——— ①

if the polynomial $\phi(D, D')$ in D, D' is not homogenous then is called a non-homogeneous LPDE.

· Hence general solution is $z = C.F + P.I$

Finding C.F.

ⓐ · Writing $\phi(D, D')$ into linear factors of the form $(D - m_1 D' - a_1)(D - m_2 D' - a_2) \cdots (D - m_n D' - a_n)z = 0$

– therefore

$$C.F = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \cdots + e^{a_n x} f_n(y + m_n x).$$

In the case of Repeated Factors e.g. $(D - mD' - a)^r z = 0$

$$z = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx) + \cdots + x^{r-1} e^{ax} f_r(y + mx).$$

Example 1.- Solve the LPDE $(D + D' - 1)(D + 2D' - 2)z = 0$

$$C.F = e^{x} f_1(y - x) + e^{2x} f_2(y - 2x).$$

Example 2 $DD'(D + 2D' + 1)z = 0$

$$(D - 0D' - 0)(0D + D' - 0)(D + 2D' + 1)z = 0$$

$$C.F = f_1(y) + f_2(x) + e^{-x} f_3(y - 2x).$$

Example 3:- $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$

$$[(D + D')(D - 2D') + 2(D + D')]z = 0$$

$$(D + D')(D - 2D' + 2)z = 0$$

$$C.F = f_1(y - x) + e^{-2x} f_2(y + 2x).$$

1.24 P.I. of non-homogeneous LPDE with constant coefficients

Consider $\phi(D,D')z = f(x,y)$

then

$$P.I. = \frac{1}{\phi(D,D')} f(x,y)$$

Case I when $f(x,y) = e^{ax+by}$ & $\phi(a,b) \neq 0$

$$P.I. = \frac{1}{\phi(D,D')} e^{ax+by}$$

$$= \frac{1}{\phi(a,b)} e^{ax+by} \quad \left\{ \begin{array}{l} \text{Replacing } D \text{ by } a \\ D' \text{ by } b \end{array} \right\}$$

Examples. Solve $s + ap + bq + abz = e^{mx+ny}$

Sol① The given equation

$$(DD' + aD + bD' + ab)z = e^{mx+ny}$$

$$(D(D'+a) + b(D'+a))z = e^{mx+ny}$$

$$(D'+a)(D+b)z = e^{mx+ny}$$

Thus, $C.F. = e^{-bx} f_1(y) + e^{-ay} f_2(x)$

$$P.I. = \frac{1}{(D+b)(D'+a)} e^{mx+ny}$$

$$= \frac{e^{mx+ny}}{(m+b)(n+a)}$$

Hence the required solution

$$z = e^{-bx} f_1(y) + e^{-ay} f_2(x) + \frac{e^{mx+ny}}{(m+b)(n+a)}$$

2: Solve $D(D-2D'-3)z = e^{x+2y}$

$$P.I. = \frac{e^{x-2y}}{1(1-2\times 2-3)} = -\frac{1}{6} e^{x+2y}.$$

Case II when $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{\phi(D,D')} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

If. $\phi(D,D') = \phi(D^2, DD', D'^2, D, D')$

$$P.I = \frac{1}{\phi(-a^2, -ab, b^2, D, D')} \sin(ax+by) \text{ or } \cos(ax+by)$$

Q. 1. Solve $(D-D'-1)(D-D'-2)z = \sin(2x+3y)$

$$P.I = \frac{1}{(D-D'-1)(D-D'-2)} \sin(2x+3y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x+3y)$$

$$= \frac{1}{-4+12-9-3D+3D'+2} \sin(2x+3y)$$

$$= \frac{1}{-(3D+3D'+1)} \sin(2x+3y)$$

$$= -\frac{(3D-3D')+1}{(3D-3D'+1)(3D-3D'-1)} \sin(2x+3y)$$

$$= -\frac{(3D-3D')+1}{9D^2+9D'^2-18DD'-1} \sin(2x+3y)$$

$$= -\frac{\{(3D-3D')+1\}}{-36-81+108-1} \sin(2x+3y)$$

$$= \frac{1}{10}(3D - 3D' + 1)\sin(2x+3y)$$

$$= \frac{1}{10}\left(6\cos(2x+3y) - 9\cos(2x+3y) + \sin(2x+3y)\right]$$

$$= \frac{1}{10}\left[\sin(2x+3y) - 3\cos(2x+3y)\right].$$

The required solution

$$z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{10}\left[\sin(2x+3y) - 3\cos(2x+3y)\right].$$

2. Solve $(D^2 - DD' + D' - 1)z = \sin(x+2y)$

Sol^{n}:- The given equ^{n}. $(D^2 - DD' + D' - 1)z = \sin(x+2y)$

$$(D-1)(D-D'+1)z = \sin(x+2y)$$

$$\text{C.F} = e^{x} f_1(y) + e^{-x} f_2(y+x)$$

$$\text{P.I} = \frac{1}{(D^2 - DD' + D' - 1)}\sin(x+2y)$$

$$= \frac{1}{-1+2+D'-1}\sin(x+2y)$$

$$= -\frac{\cos(x+2y)}{2}$$

$$z = e^{x} f_1(y) + e^{-x} f_2(y+x) - \frac{\cos(x+2y)}{2}.$$

Case III. when $f(x,y) = x^m y^n$

$$P.I = \frac{1}{\phi(D,D')} x^m y^n$$

(a). when $m > n$, expanding powers of $\frac{D'}{D}$

(b) when $n < m$, expanding powers of $\frac{D}{D'}$

Q.1. Solve LPDE $(D-D'-1)(D-D'-2)z = e^{3x-y} + x$

$$C.F = e^{x} f_1(y+x) + e^{2x} f_2(y+2x)$$

$$P.I = \frac{1}{(D-D'-1)(D-D'-2)} \left(e^{3x-y} + x\right)$$

$$= \frac{1}{(3+1-1)(3+1-2)} e^{3x-y} + \frac{1}{(1-D+D')(2-D+D')} (x)$$

$$= \frac{1}{6} e^{3x-y} + \frac{1}{2} [1-(D-D')]^{-1} \left[1 - \frac{(D-D')}{2}\right]^{-1} x$$

$$= \frac{e^{3x-y}}{6} + \frac{1}{2}\left[(1+D-D')\left[1+\frac{D-D'}{2}\right]\right] x$$

$$= \frac{e^{3x-y}}{6} + \frac{1}{2}\left[x+1+\frac{1}{2}+0\right]$$

$$= \frac{e^{3x-y}}{6} + \frac{1}{2}\left(x+\frac{3}{2}\right)$$

Hence required solution

$$z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6} e^{3x-y} + \frac{1}{2}\left(x+\frac{3}{2}\right).$$

Q.2. Solve $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^m y^n$

Sol^n:- The given equation $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^m y^n$ — (1)

Putting $xD = D_1$, $x^2D^2 = D_1(D_1-1)$, $yD = D_1'$, $y^2D^2 = D_1'(D_1'-1)$ into (1).

$$\{D_1(D_1-1) + 2D_1D_1' + D_1'(D_1'-1)\}z = e^{mX+nY}$$

$$[(D_1^2 + 2D_1D_1' + D_1'^2) - (D_1+D_1')]z = e^{mX+nY}$$

$$[(D_1+D_1')^2 - (D_1+D_1')]z = e^{mX+nY}$$

$$(D_1+D_1')(D+D_1'-1)z = e^{mX+nY}$$

$$C.F = f_1(Y-X) + e^X f_2(Y-X)$$

$$= g_1\left(\log\frac{y}{x}\right) + x g_2\left(\log\frac{y}{x}\right)$$

$$= g_1\left(\frac{y}{x}\right) + x\, g_2\left(\frac{y}{x}\right)$$

Now

$$P.I = \frac{1}{(D_1+D_1')(D_1+D_1'-1)} e^{mX+nY}$$

$$= \frac{1}{(m+n)(m+n-1)} \cdot e^{mX+nY} = \frac{x^m y^n}{(m+n)(m+n-1)}$$

Hence general solution is

$$z = g_1\left(\frac{y}{x}\right) + g_2\left(\frac{y}{x}\right) + \frac{x^m y^n}{(m+n)(m+n+1)}$$

Q. Solve $s + p - q = z + xy$.

<u>Case IV</u> when $f(x,y) = e^{ax+by} \cdot V$

where V is the function of x & y.

$$P.I = \frac{1}{\phi(D, D')} e^{ax+by} \cdot V = e^{ax+by} \frac{1}{\phi(D+a, D'+b)} \cdot V$$

V can be either. (i) $e^{a_1x+b_1y}$

(ii) $\sin(ax+by)$ or $\cos(ax+by)$

(iii) $x^m y^n$

(iv) Constant (say 1, 2, ...).

Q.1. Solve $(D - 3D' - 2)^3 z = 6e^{2x} \sin 3x$

Soln: $C.f = e^{2x}\{f_1(y+3x) + xf_2(y+3x) + x^2 f_3(y+3x)\}$

$$P.I = \frac{1}{(D-3D'-2)^3} 6e^{2x} \sin(3x+y)$$

$$= 6e^{2x} \frac{1}{(D+\cancel{2}-3D'-\cancel{2})^3} \sin(3x+y)$$

$$= 6e^{2x} \frac{1}{(D-3D')^3} \sin(3x+y)$$

$$= 6e^{2x} \frac{x^3}{3!} \sin(3x+y)$$

$$= \cancel{6}e^{2x} \frac{x^3}{\cancel{6}} \sin(3x+y)$$

$$z = e^{2x}\{f_1(y+3x) + xf_2(y+3x) + x^2 f_3(y+3x)\} + e^{2x} x^3 \sin(3x+y).$$

Q. Solve $r - 4s + 4t - p - 2q = e^{x-y}$

C.F $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$

$$[(D-2D')^2 + (D-2D')]z = e^{x+y}$$

$$(D-2D')(D-2D'+1)z = e^{x+y}$$

$$C.F = f_1(y+2x) + e^{-x} f_2(y+2x)$$

$$P.I = \frac{1}{(D-2D'+1)}\left\{\frac{1}{(D-2D')} e^{x-y}\right\}$$

$$= \frac{1}{D-2D'+1}(-1)\int e^{u}\,du$$

$$= -\frac{1}{D-2D'+1} e^{x+y}$$

$$= e^{x+y}\frac{(-1)}{(D+1)-2(D'-1)+1}(1)$$

$$= \frac{-e^{x+y}}{D-2D'}$$

$$= \frac{-e^{x+y}}{D\left(1-\frac{2D'}{D}\right)} 1$$

$$= \frac{-e^{x+y}}{D}\left(1+\frac{2D'}{D}\right) 1$$

$$= -xe^{x+y}$$

$$z = f_1(y+2x) + e^{-x} f_2(x+2x) - xe^{x+y}$$

Equation Reducible to PDE with Constant Coefficient

- Consider Euler-cauchy type equation

$$(a_0 x^n D^n + a_1 x^{n-1} y D^{n-1} D' + a_2 x^{n-2} D^{n-2} D'^2 + \dots + a_n y^n D'^n + \dots) z = f(x,y) \quad \text{--- (1)}$$

~~StepI~~ ~~Let~~ where $D \equiv \frac{\partial}{\partial x}$ & $D' \equiv \frac{\partial}{\partial y}$

step1: let $x = e^X$, $y = e^Y$

$X = \log x$, $Y = \log y$

step2: we have $D \equiv \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$. Also let

$D_1 = \frac{\partial}{\partial X}$, $D_1' = \frac{\partial}{\partial Y}$

Step3:- $xD = D_1$, $x^2 D^2 = D_1(D_1 - 1)$, $x^3 D^3 = D_1(D_1 - 1)(D_1 - 2) \dots$

$yD_p' = D_1'$, $y^2 D^2 = D_1'(D_1' - 1)$, $y^3 D^3 = D_1'(D_1' - 1)(D_1' - 2) \dots$

and so on

Step4:- Eq (1) becomes a ~~Homogeneous~~ ~~LPDE as~~:

$$(b_0 D_1^n + b_1 D_1^{n-1} D_1' + b_2 D_1^{n-2} D_1'^2 + \dots + b_n D_1'^m) z = f(x,y) \quad \text{--- (2)}$$

equation (2) Homogeneous LPDE or Non-Homogeneous LPDE.

step5: Using $X = \log x$ & $Y = \log y$.

Example1. solve the linear partial DE

$$x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4.$$

Sol^n:- The given equation

$$(x^2 D^2 - 4xy DD' + 4y^2 D'^2 + 6yD') z = x^3 y^4 \quad \text{--- (1)}$$

Let $x = e^X$ & $y = e^Y$.

$X = \log x$ & $Y = \log y$,

~~subst~~ Putting

$xD = D_1$, $x^2 D^2 = D_1(D_1 - 1)$

$yD' = D_1'$, $y^2 D'^2 = D_1'(D_1' - 1)$

from (1).

$$(D_1(D_1 - 1) - 4 D_1 D_1' + 4 D_1'(D_1' - 1) + 6 D_1') z = e^{3X + 4Y}$$

$$[D_1^2 - D_1 - \cancel{4D_1D_1'} + \cancel{4D_1D_1'^2} + 4D_1' + 6D_1']z = e^{3x+4y}$$ (38)

$$[(D_1^2 - 4D_1D_1' + 4D_1'^2) - (D - 2D')]\, z = e^{3x+4y}$$

$$[(D_1 - 2D_1')^2 - (D - 2D')]\, z = e^{3x+4y}$$

$$(D_1 - 2D_1')(D_1 - 2D' - 1)\, z = e^{3x+4y}$$

So, $$C.f = f_1(Y + 2X) + e^X f_2(Y + 2X)$$

$$= f_1(\log y + 2\log x) + e^{\log x} f_2(\log y + 2\log x)$$

$$P.I = \frac{1}{(D_1 - 2D_1' - 1)}\left[\frac{1}{D_1 - 2D_1'} e^{3X+4Y}\right]$$

$$= \frac{1}{(D_1 - 2D_1' - 1)}\left[\frac{-1}{5}\int e^u du\right] \quad \text{where } u = 3x + 4y$$

$$= \frac{1}{(D_1 - 2D_1' - 1)}\left(\frac{-1}{5} e^{3x+4y}\right)$$

$$= \frac{-1}{5}\left[\frac{-1}{6} e^{3x+4y}\right]$$

$$= \frac{1}{30} e^{3x+4y}$$

Hence the required solution is

$$z = C.f + P.I = f_1(\log yx^2) + x f_2(\log xy^2) + \frac{1}{30}x^3y^3$$

$$z = g_1(yx^2) + x\, g_2(xy^2) + \frac{1}{30}x^3y^3$$